GRE®

GRADUATE RECORD EXAMINATIONS®

Math Review
Large Print (18 point) Edition
Chapter 2: Algebra

The GRE® Math Review consists of 4 chapters: Arithmetic, Algebra, Geometry, and Data Analysis. This is the Large Print edition of the Algebra Chapter of the Math Review. Downloadable versions of large print (PDF) and accessible electronic format (Word) of each of the 4 chapters of the Math Review, as well as a Large Print Figure supplement for each chapter are available from the GRE® website. Other downloadable practice and test familiarization materials in large print and accessible electronic formats are also available. Tactile figure supplements for the 4 chapters of the Math Review, along with additional accessible practice and test familiarization materials in other formats, are available from ETS Disability Services Monday to Friday 8:30 a.m. to 5 p.m. New York time, at 1-609-771-7780, or 1-866-387-8602 (toll free for test takers in the United States, U.S. Territories, and Canada), or via email at stassd@ets.org.

The mathematical content covered in this edition of the Math Review is the same as the content covered in the standard edition of the Math Review. However, there are differences in the presentation of some of the material. These differences are the result of adaptations made for presentation of the material in accessible formats. There are also slight differences between the various accessible formats, also as a result of specific adaptations made for each format.

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Overview of the Math Review

The Math Review consists of 4 chapters: Arithmetic, Algebra, Geometry, and Data Analysis.

Each of the 4 chapters in the Math Review will familiarize you with the mathematical skills and concepts that are important to understand in order to solve problems and reason quantitatively on the Quantitative Reasoning measure of the GRE® revised General Test.

The material in the Math Review includes many definitions, properties, and examples, as well as a set of exercises (with answers) at the end of each chapter. Note, however that this review is not intended to be all-inclusive—there may be some concepts on the test that are not explicitly presented in this review. If any topics in this review seem especially unfamiliar or are covered too briefly, we encourage you to consult appropriate mathematics texts for a more detailed treatment.

Overview of this Chapter

Basic algebra can be viewed as an extension of arithmetic. The main concept that distinguishes algebra from arithmetic is that of a **variable**, which is a letter that represents a quantity whose value is unknown. The letters x and y are often used as variables, although any letter can be used. Variables enable you to present a word problem in terms of unknown quantities by using algebraic expressions, equations, inequalities, and functions. This chapter reviews these algebraic tools and then progresses to several examples of applying them to solve real-life word problems. The chapter ends with coordinate geometry and graphs of functions as other important algebraic tools for solving problems.

2.1 Operations with Algebraic Expressions

An **algebraic expression** has one or more variables and can be written as a single **term** or as a sum of terms. Here are four examples of algebraic expressions.

Example A: $2x$

Example B: $y - \frac{1}{4}$

Example C: $w^3z + 5z^2 - z^2 + 6$

Example D: $\frac{8}{n + p}$

In the examples above, $2x$ is a single term, $y - \frac{1}{4}$ has two terms, $w^3z + 5z^2 - z^2 + 6$ has four terms, and $\frac{8}{n + p}$ has one term. In the expression $w^3z + 5z^2 - z^2 + 6$, the terms $5z^2$ and $-z^2$ are called **like terms** because they have the same variables, and the corresponding variables have the same exponents. A term that has no variable is called a **constant** term. A number that is multiplied by variables is called the **coefficient** of a term. For example, in the expression $2x^2 + 7x - 5$, 2 is the coefficient of the term $2x^2$, 7 is the coefficient of the term $7x$, and -5 is a constant term.

The same rules that govern operations with numbers apply to operations with algebraic expressions. One additional rule, which helps in simplifying algebraic expressions, is that like terms can be combined by simply adding their coefficients, as the following three examples show.

Example A: $2x + 5x = 7x$

Example B: $w^3z + 5z^2 - z^2 + 6 = w^3z + 4z^2 + 6$

Example C: $3xy + 2x - xy - 3x = 2xy - x$

A number or variable that is a factor of each term in an algebraic expression can be factored out, as the following three examples show.

Example A: $4x + 12 = 4(x + 3)$

Example B: $15y^2 - 9y = 3y(5y - 3)$

Example C: The expression $\dfrac{7x^2 + 14x}{2x + 4}$ can be simplified as follows.

First factor the numerator and the denominator to get $\dfrac{7x(x + 2)}{2(x + 2)}$.

Now, since $x + 2$ occurs in both the numerator and the denominator, it can be canceled out when $x + 2 \neq 0$, that is, when $x \neq -2$ (since division by 0 is not defined). Therefore, for all $x \neq -2$, the expression is equivalent to $\dfrac{7x}{2}$.

To multiply two algebraic expressions, each term of the first expression is multiplied by each term of the second expression, and the results are added, as the following example shows.

To multiply

$(x + 2)(3x - 7)$

first multiply each term of the expression $x + 2$ by each term of the expression $3x - 7$ to get the expression

$x(3x) + x(-7) + 2(3x) + 2(-7)$.

Then multiply each term to get

$3x^2 - 7x + 6x - 14$.

Finally, combine like terms to get

$3x^2 - x - 14$.

So you can conclude that $(x + 2)(3x - 7) = 3x^2 - x - 14$.

A statement of equality between two algebraic expressions that is true for all possible values of the variables involved is called an **identity**. All of the statements above are identities. Here are three standard identities that are useful.

Identity 1: $(a + b)^2 = a^2 + 2ab + b^2$

Identity 2: $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Identity 3: $a^2 - b^2 = (a + b)(a - b)$

All of the identities above can be used to modify and simplify algebraic expressions. For example, identity 3, $a^2 - b^2 = (a + b)(a - b)$, can be used to simplify the algebraic expression $\frac{x^2 - 9}{4x - 12}$ as follows.

$$\frac{x^2 - 9}{4x - 12} = \frac{(x + 3)(x - 3)}{4(x - 3)}.$$

Now, since $x - 3$ occurs in both the numerator and the denominator, it can be canceled out when $x - 3 \neq 0$, that is, when $x \neq 3$ (since division by 0 is not defined). Therefore, for all $x \neq 3$, the expression is equivalent to $\frac{x + 3}{4}$.

A statement of equality between two algebraic expressions that is true for only certain values of the variables involved is called an **<u>equation</u>**. The values are called the **<u>solutions</u>** of the equation.

The following are three basic types of equations.

Type 1: A **linear equation in one variable**: for example, $3x + 5 = -2$

Type 2: A **linear equation in two variables**: for example, $x - 3y = 10$

Type 3: A **quadratic equation in one variable**: for example, $20y^2 + 6y - 17 = 0$

2.2 Rules of Exponents

In the algebraic expression x^a, where x is raised to the power a, x is called a **base** and a is called an **exponent**. Here are seven basic rules of exponents, where the bases x and y are nonzero real numbers and the exponents a and b are integers.

Rule 1: $x^{-a} = \frac{1}{x^a}$

Example A: $4^{-3} = \frac{1}{4^3} = \frac{1}{64}$

Example B: $x^{-10} = \dfrac{1}{x^{10}}$

Example C: $\dfrac{1}{2^{-a}} = 2^{a}$

Rule 2: $\left(x^{a}\right)\left(x^{b}\right) = x^{a+b}$

Example A: $\left(3^{2}\right)\left(3^{4}\right) = 3^{2+4} = 3^{6} = 729$

Example B: $\left(y^{3}\right)\left(y^{-1}\right) = y^{2}$

Rule 3: $\dfrac{x^{a}}{x^{b}} = x^{a-b} = \dfrac{1}{x^{b-a}}$

Example A: $\dfrac{5^{7}}{5^{4}} = 5^{7-4} = 5^{3} = 125$

Example B: $\dfrac{t^{3}}{t^{8}} = t^{-5} = \dfrac{1}{t^{5}}$

Rule 4: $x^0 = 1$

Example A: $7^0 = 1$

Example B: $(-3)^0 = 1$

Note that 0^0 is not defined.

Rule 5: $\left(x^a\right)\left(y^a\right) = (xy)^a$

Example A: $\left(2^3\right)\left(3^3\right) = 6^3 = 216$

Example B: $(10z)^3 = 10^3 z^3 = 1{,}000z^3$

Rule 6: $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$

Example A: $\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16}$

Example B: $\left(\frac{r}{4t}\right)^3 = \frac{r^3}{64t^3}$

Rule 7: $\left(x^a\right)^b = x^{ab}$

Example A: $\left(2^5\right)^2 = 2^{10} = 1{,}024$

Example B: $\left(3y^6\right)^2 = \left(3^2\right)\left(y^6\right)^2 = 9y^{12}$

The rules above are identities that are used to simplify expressions. Sometimes algebraic expressions look like they can be simplified in similar ways, but in fact they cannot. In order to avoid mistakes commonly made when dealing with exponents keep the following six cases in mind.

Case 1: $x^a y^b \neq (xy)^{a+b}$

Note that in the expression $x^a y^b$ the bases are not the same, so rule 2, $\left(x^a\right)\left(x^b\right) = x^{a+b}$, does not apply.

Case 2: $\left(x^a\right)^b \neq x^a x^b$

Instead, $\left(x^a\right)^b = x^{ab}$ and $x^a x^b = x^{a+b}$; for example, $\left(4^2\right)^3 = 4^6$ and $4^2 4^3 = 4^5$.

Case 3: $(x + y)^a \neq x^a + y^a$

Recall that $(x + y)^2 = x^2 + 2xy + y^2$; that is, the correct expansion contains terms such as $2xy$.

Case 4: $(-x)^2 \neq -x^2$

Instead, $(-x)^2 = x^2$. Note carefully where each minus sign appears.

Case 5: $\sqrt{x^2 + y^2} \neq x + y$

Case 6: $\dfrac{a}{x + y} \neq \dfrac{a}{x} + \dfrac{a}{y}$

But it is true that $\dfrac{x + y}{a} = \dfrac{x}{a} + \dfrac{y}{a}$.

2.3 Solving Linear Equations

To **solve an equation** means to find the values of the variables that make the equation true; that is, the values that **satisfy the equation**. Two equations that have the same solutions are called **equivalent equations**. For example, $x + 1 = 2$ and $2x + 2 = 4$ are equivalent equations; both are true when $x = 1$ and are false otherwise. The general method for solving an equation is to find successively simpler equivalent equations so that the simplest equivalent equation makes the solutions obvious.

The following two rules are important for producing equivalent equations.

Rule 1: When the same constant is added to or subtracted from both sides of an equation, the equality is preserved and the new equation is equivalent to the original equation.

Rule 2: When both sides of an equation are multiplied or divided by the same nonzero constant, the equality is preserved and the new equation is equivalent to the original equation.

A **linear equation** is an equation involving one or more variables in which each term in the equation is either a constant term or a variable multiplied by a coefficient. None of the variables are multiplied together or raised to a power greater than 1. For example, $2x + 1 = 7x$ and $10x - 9y - z = 3$ are linear equations, but $x + y^2 = 0$ and $xz = 3$ are not.

Linear Equations in One Variable

To solve a linear equation in one variable, simplify each side of the equation by combining like terms. Then use the rules for producing simpler equivalent equations.

Example 2.3.1: Solve the equation
$11x - 4 - 8x = 2(x + 4) - 2x$ as follows.

Combine like terms to get	$3x - 4 = 2x + 8 - 2x$
Simplify the right side to get	$3x - 4 = 8$
Add 4 to both sides to get	$3x - 4 + 4 = 8 + 4$
Divide both sides by 3 to get	$\frac{3x}{3} = \frac{12}{3}$
Simplify to get	$x = 4$

You can always check your solution by substituting it into the original equation.

Note that it is possible for a linear equation to have no solutions. For example, the equation $2x + 3 = 2(7 + x)$ has no solution, since it is equivalent to the equation $3 = 14$, which is false. Also, it is possible that what looks to be a linear equation turns out to be an identity when you try to solve it. For example, $3x - 6 = -3(2 - x)$ is true for all values of x, so it is an identity.

Linear Equations in Two Variables

A linear equation in two variables, x and y, can be written in the form

$$ax + by = c,$$

where a, b, and c are real numbers and a and b are not both zero. For example, $3x + 2y = 8$ is a linear equation in two variables.

A solution of such an equation is an **ordered pair** of numbers (x, y) that makes the equation true when the values of x and y are substituted into the equation. For example, both $(2, 1)$ and $\left(-\frac{2}{3}, 5\right)$ are solutions of the equation $3x + 2y = 8$, but $(1, 2)$ is not a solution. A linear equation in two variables has infinitely many solutions. If another linear equation in the same variables is given, it may be possible to find a unique solution of both equations. Two equations with the same variables are called a **system of equations**, and the equations in the system are called **simultaneous equations**. To solve a system of two equations means to find an ordered pair of numbers that satisfies both equations in the system.

There are two basic methods for solving systems of linear equations, by **substitution** or by **elimination**. In the substitution method, one equation is manipulated to express one variable in terms of the other. Then the expression is substituted in the other equation.

For example, to solve the system of two equations

$$4x + 3y = 13$$
$$x + 2y = 2$$

you can express x in the second equation in terms of y as $x = 2 - 2y$.

Then substitute $2 - 2y$ for x in the first equation to find the value of y.

The value of y can be found as follows.

Substitute for x in the first equation to get $4(2 - 2y) + 3y = 13$

Multiply out the first term and get $8 - 8y + 3y = 13$

Subtract 8 from both sides to get $-8y + 3y = 5$

Combine like terms to get $-5y = 5$

Divide both sides by -5 to get $y = -1$

Then -1 can be substituted for y in either equation to find the value of x.

We use the second equation as follows.

Substitute for y in the second equation to get $x + 2(-1) = 2$.

That is, $x - 2 = 2$.

Add 2 to both sides to get $x = 4$.

In the elimination method, the object is to make the coefficients of one variable the same in both equations so that one variable can be eliminated either by adding the equations together or by subtracting one from the other. In the example above, multiplying both sides of the second equation, $x + 2y = 2$, by 4 yields $4(x + 2y) = 4(2)$, or $4x + 8y = 8$. Now you have two equations with the same coefficient of x.

$$\begin{aligned} 4x + 3y &= 13 \\ 4x + 8y &= 8 \end{aligned}$$

If you subtract the equation $4x + 8y = 8$ from the equation $4x + 3y = 13$, the result is $-5y = 5$. Thus, $y = -1$, and substituting -1 for y in either of the original equations yields $x = 4$.

By either method, the solution of the system is $x = 4$ and $y = -1$, or $(x, y) = (4, -1)$.

2.4 Solving Quadratic Equations

A **quadratic equation** in the variable x is an equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b, and c are real numbers and $a \neq 0$. When such an equation has solutions, they can be found using the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where the notation $\pm$ is shorthand for indicating two solutions—one that uses the plus sign and the other that uses the minus sign.

Example 2.4.1: In the quadratic equation $2x^2 - x - 6 = 0$, we have $a = 2$, $b = -1$, and $c = -6$. Therefore, the quadratic formula yields

$$
\begin{aligned}
x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-6)}}{2(2)} \\
&= \frac{1 \pm \sqrt{49}}{4} \\
&= \frac{1 \pm 7}{4}
\end{aligned}
$$

Hence the two solutions are $x = \frac{1+7}{4} = 2$ and $x = \frac{1-7}{4} = -\frac{3}{2}$.

Quadratic equations have at most two real solutions, as in example 2.4.1 above. However, some quadratic equations have only one real solution. For example, the quadratic equation $x^2 + 4x + 4 = 0$ has only one solution, which is $x = -2$. In this case, the expression under the square root symbol in the quadratic formula is equal to 0, and so adding or subtracting 0 yields the same result. Other quadratic equations have no real solutions; for example, $x^2 + x + 5 = 0$. In this case, the expression under the square root symbol is negative, so the entire expression is not a real number.

Some quadratic equations can be solved more quickly by factoring. For example, the quadratic equation $2x^2 - x - 6 = 0$ in example 2.4.1 can be factored as $(2x + 3)(x - 2) = 0$. When a product is equal to 0, at least one of the factors must be equal to 0, so either $2x + 3 = 0$ or $x - 2 = 0$.

If $2x + 3 = 0$, then $2x = -3$ and $x = -\frac{3}{2}$.

If $x - 2 = 0$, then $x = 2$.

Thus the solutions are $-\frac{3}{2}$ and 2.

Example 2.4.2: The quadratic equation $5x^2 + 3x - 2 = 0$ can be easily factored as $(5x - 2)(x + 1) = 0$.

Therefore, either $5x - 2 = 0$ or $x + 1 = 0$.

If $5x - 2 = 0$, then $x = \frac{2}{5}$.

If $x + 1 = 0$, then $x = -1$.

Thus the solutions are $\frac{2}{5}$ and -1.

2.5 Solving Linear Inequalities

A mathematical statement that uses one of the following inequality signs is called an **<u>inequality</u>**.

$<$	less than
$>$	greater than
$\leq$	less than or equal to
$\geq$	greater than or equal to

Inequalities can involve variables and are similar to equations, except that the two sides are related by one of the inequality signs instead of the equality sign used in equations. For example, the inequality $4x - 1 \leq 7$ is a linear inequality in one variable, which states that "$4x - 1$ is less than or equal to 7." To **solve an inequality** means to find the set of all values of the variable that make the inequality true. This set of values is also known as the **solution set** of an inequality. Two inequalities that have the same solution set are called **equivalent inequalities**.

The procedure used to solve a linear inequality is similar to that used to solve a linear equation, which is to simplify the inequality by isolating the variable on one side of the inequality, using the following two rules.

Rule 1: When the same constant is added to or subtracted from both sides of an inequality, the direction of the inequality is preserved and the new inequality is equivalent to the original.

Rule 2: When both sides of the inequality are multiplied or divided by the same nonzero constant, the direction of the inequality is ***preserved if the constant is positive*** but the direction is ***reversed if the constant is negative***. In either case, the new inequality is equivalent to the original.

Example 2.5.1: The inequality $-3x + 5 \le 17$ can be solved as follows.

Subtract 5 from both sides to get $-3x \le 12$

Divide both sides by -3 and reverse the direction of the inequality to get $\frac{-3x}{-3} \ge \frac{12}{-3}$

That is, $x \ge -4$

Therefore, the solution set of $-3x + 5 \le 17$ consists of all real numbers greater than or equal to -4.

Example 2.5.2: The inequality $\frac{4x + 9}{11} < 5$ can be solved as follows.

Multiply both sides by 11 to get $4x + 9 < 55$

Subtract 9 from both sides to get $4x < 46$

Divide both sides by 4 to get $x < \frac{46}{4}$

That is, $x < 11.5$

Therefore, the solution set of $\frac{4x + 9}{11} < 5$ consists of all real numbers less than 11.5.

2.6 Functions

An algebraic expression in one variable can be used to define a **function** of that variable. Functions are usually denoted by letters such as f, g, and h. For example, the algebraic expression $3x + 5$ can be used to define a function f by

$f(x) = 3x + 5,$

where $f(x)$ is called the value of f at x and is obtained by substituting the value of x in the expression above. For example, if $x = 1$ is substituted in the expression above, the result is $f(1) = 8.$

It might be helpful to think of a function f as a machine that takes an input, which is a value of the variable x, and produces the corresponding output, $f(x)$. For any function, each input x gives exactly one output $f(x)$. However, more than one value of x can give the same output $f(x)$. For example, if g is the function defined by $g(x) = x^2 - 2x + 3,$ then $g(0) = 3$ and $g(2) = 3.$

The **domain** of a function is the set of all permissible inputs, that is, all permissible values of the variable x. For the functions f and g defined above, the domain is the set of all real numbers. Sometimes the domain of the function is given explicitly and is restricted to a specific set of values of x. For example, we can define the function h by $h(x) = x^2 - 4$ for $-2 \leq x \leq 2.$

Without an explicit restriction, the domain is assumed to be the set of all values of x for which $f(x)$ is a real number.

Example 2.6.1: Let f be the function defined by $f(x) = \frac{2x}{x - 6}$. In this case, f is not defined at $x = 6$ because $\frac{12}{0}$ is not defined. Hence, the domain of f consists of all real numbers except for 6.

Example 2.6.2: Let g be the function defined by $g(x) = x^3 + \sqrt{x + 2} - 10$. In this case, $g(x)$ is not a real number if $x < -2$. Hence, the domain of g consists of all real numbers x such that $x \geq -2$.

Example 2.6.3: Let h be the function defined by $h(x) = |x|$, the **absolute value** of x, which is the distance between x and 0 on the number line (see Chapter 1: Arithmetic, Section 1.5). The domain of h is the set of all real numbers. Also, $h(x) = h(-x)$ for all real numbers x, which reflects the property that on the number line the distance between x and 0 is the same as the distance between $-x$ and 0.

2.7 Applications

Translating verbal descriptions into algebraic expressions is an essential initial step in solving word problems. Three examples of verbal descriptions and their translations are given below.

Example A: If the square of the number x is multiplied by 3, and then 10 is added to that product, the result can be represented algebraically by $3x^2 + 10$.

Example B: If John's present salary s is increased by 14 percent, then his new salary can be represented algebraically by $1.14s$.

Example C: If y gallons of syrup are to be distributed among 5 people so that one particular person gets 1 gallon and the rest of the syrup is divided equally among the remaining 4, then the number of gallons of syrup each of those 4 people will get can be represented algebraically by $\frac{y-1}{4}$.

The remainder of this section gives examples of various applications.

Applications Involving Average, Mixture, Rate, and Work Problems

Example 2.7.1: Ellen has received the following scores on 3 exams: 82, 74, and 90. What score will Ellen need to receive on the next exam so that the average (arithmetic mean) score for the 4 exams will be 85 ?

Solution: Let x represent the score on Ellen's next exam. This initial step of assigning a variable to the quantity that is sought is an important beginning to solving the problem. Then in terms of x, the average of the 4 exams is

$$\frac{82 + 74 + 90 + x}{4},$$

which is supposed to equal 85. Now simplify the expression and set it equal to 85:

$$\frac{82 + 74 + 90 + x}{4} = \frac{246 + x}{4} = 85.$$

Solving the resulting linear equation for x, you get
$246 + x = 340$, and $x = 94$.

Therefore, Ellen will need to attain a score of 94 on the next exam.

Example 2.7.2: A mixture of 12 ounces of vinegar and oil is 40 percent vinegar, where all of the measurements are by weight. How many ounces of oil must be added to the mixture to produce a new mixture that is only 25 percent vinegar?

Solution: Let x represent the number of ounces of oil to be added. Then the total number of ounces of the new mixture will be $12 + x$, and the total number of ounces of vinegar in the new mixture will be $(0.40)(12)$. Since the new mixture must be 25 percent vinegar,

$$\frac{(0.40)(12)}{12 + x} = 0.25.$$

Therefore $(0.40)(12) = (12 + x)(0.25)$.

Multiplying out gives $4.8 = 3 + 0.25x$, so $1.8 = 0.25x$ and $7.2 = x$.

Thus, 7.2 ounces of oil must be added to produce a new mixture that is 25 percent vinegar.

Example 2.7.3: In a driving competition, Jeff and Dennis drove the same course at average speeds of 51 miles per hour and 54 miles per hour, respectively. If it took Jeff 40 minutes to drive the course, how long did it take Dennis?

Solution: Let x be the time, in minutes, that it took Dennis to drive the course. The distance d, in miles, is equal to the product of the rate r, in miles per hour, and the time t, in hours; that is,

$$d = rt.$$

Note that since the rates are given in miles per ***hour***, it is necessary to express the times in hours; for example, 40 minutes equals $\frac{40}{60}$ of an hour. Thus, the distance traveled by Jeff is the product of his speed and his time, $(51)\left(\frac{40}{60}\right)$ miles, and the distance traveled by Dennis is similarly represented by $(54)\left(\frac{x}{60}\right)$ miles. Since the distances are equal, it follows that $(51)\left(\frac{40}{60}\right) = (54)\left(\frac{x}{60}\right)$.

From this equation it follows that $(51)(40) = 54x$ and $x = \frac{(51)(40)}{54} \approx 37.8$.

Thus, it took Dennis approximately 37.8 minutes to drive the course.

Example 2.7.4: Working alone at its constant rate, machine A takes 3 hours to produce a batch of identical computer parts. Working alone at its constant rate, machine B takes 2 hours to produce an identical batch of parts. How long will it take the two machines, working simultaneously at their respective constant rates, to produce an identical batch of parts?

Solution: Since machine A takes 3 hours to produce a batch, machine A can produce $\frac{1}{3}$ of the batch in 1 hour. Similarly, machine B can produce $\frac{1}{2}$ of the batch in 1 hour. If we let x represent the number of hours it takes both machines, working simultaneously, to produce the batch, then the two machines will produce $\frac{1}{x}$ of the job in 1 hour. When the two machines work together, adding their individual production rates, $\frac{1}{3}$ and $\frac{1}{2}$, gives their combined production rate $\frac{1}{x}$. Therefore, it follows that $\frac{1}{3} + \frac{1}{2} = \frac{1}{x}$. This equation is equivalent to $\frac{2}{6} + \frac{3}{6} = \frac{1}{x}$. So $\frac{5}{6} = \frac{1}{x}$ and $\frac{6}{5} = x$.

Thus, working together, the machines will take $\frac{6}{5}$ hours, or 1 hour 12 minutes, to produce a batch of parts.

Example 2.7.5: At a fruit stand, apples can be purchased for \$0.15 each and pears for \$0.20 each. At these rates, a bag of apples and pears was purchased for \$3.80. If the bag contained 21 pieces of fruit, how many of the pieces were pears?

Solution: If a represents the number of apples purchased and p represents the number of pears purchased, the information can be translated into the following system of two equations.

Total cost equation: $0.15a + 0.20p = 3.80$

Total number of fruit equation: $a + p = 21$

From the total number of fruit equation, $a = 21 - p$. Substituting $21 - p$ into the total cost equation for a gives the equation $0.15(21 - p) + 0.20p = 3.80$.

So,

$(0.15)(21) - 0.15p + 0.20p = 3.80,$

which is equivalent to

$3.15 - 0.15p + 0.20p = 3.80.$

Therefore,

$0.05p = 0.65$, and $p = 13.$

Thus, of the 21 pieces of fruit, 13 were pears.

Example 2.7.6: To produce a particular radio model, it costs a manufacturer \$30 per radio, and it is assumed that if 500 radios are produced, all of them will be sold. What must be the selling price per radio to ensure that the profit (revenue from the sales minus the total production cost) on the 500 radios is greater than \$8,200 ?

Solution: If y represents the selling price per radio, then the profit is $500(y - 30)$. Therefore, $500(y - 30) > 8{,}200$.

Multiplying out gives $500y - 15,000 > 8,200,$ which simplifies to $500y > 23,200$ and then to $y > 46.4$. Thus, the selling price must be greater than \$46.40 to ensure that the profit is greater than \$8,200.

Applications Involving Interest

Some applications involve computing **interest** earned on an investment during a specified time period. The interest can be computed as simple interest or compound interest.

Simple interest is based only on the initial deposit, which serves as the amount on which interest is computed, called the **principal**, for the entire time period. If the amount P is invested at a ***simple annual interest rate of r percent***, then the value V of the investment at the end of t years is given by the formula

$$V = P\left(1 + \frac{rt}{100}\right),$$

where P and V are in dollars.

In the case of **compound interest**, interest is added to the principal at regular time intervals, such as annually, quarterly, and monthly. Each time interest is added to the principal, the interest is said to be compounded. After each compounding, interest is earned on the new principal, which is the sum of the preceding principal and the interest just added. If the amount *P* is invested at an ***annual interest rate of r percent***, ***compounded annually***, then the value *V* of the investment at the end of *t* years is given by the formula

$$V = P\left(1 + \frac{r}{100}\right)^{t}.$$

If the amount *P* is invested at an ***annual interest rate of r percent***, ***compounded n times per year***, then the value *V* of the investment at the end of *t* years is given by the formula

$$V = P\left(1 + \frac{r}{100n}\right)^{nt}.$$

Example 2.7.7: If \$10,000 is invested at a simple annual interest rate of 6 percent, what is the value of the investment after half a year?

Solution: According to the formula for simple interest, the value of the investment after $\frac{1}{2}$ year is

$$\$10{,}000\left(1 + 0.06\left(\frac{1}{2}\right)\right) = \$10{,}000(1.03) = \$10{,}300.$$

Example 2.7.8: If an amount P is to be invested at an annual interest rate of 3.5 percent, compounded annually, what should be the value of P so that the value of the investment is \$1,000 at the end of 3 years?

Solution: According to the formula for 3.5 percent annual interest, compounded annually, the value of the investment after 3 years is

$$P(1 + 0.035)^3,$$

and we set it to be equal to \$1,000

$$P(1 + 0.035)^3 = \$1,000.$$

To find the value of *P*, we divide both sides of the equation by $(1 + 0.035)^3$.

$$P = \frac{\$1,000}{(1 + 0.035)^3} \approx \$901.94.$$

Thus, an amount of approximately $901.94 should be invested.

Example 2.7.9: A college student expects to earn at least $1,000 in interest on an initial investment of $20,000. If the money is invested for one year at interest compounded quarterly, what is the least annual interest rate that would achieve the goal?

Solution: According to the formula for r percent annual interest, compounded quarterly, the value of the investment after 1 year is

$$\$20,000\left(1+\frac{r}{400}\right)^4.$$

By setting this value greater than or equal to $21,000 and solving for r, we get $\$20,000\left(1+\frac{r}{400}\right)^4 \geq \$21,000$, which simplifies to $\left(1+\frac{r}{400}\right)^4 \geq 1.05$.

Recall that taking the positive fourth root of each side of an inequality preserves the direction of the inequality. (It is also true that taking the positive square root or any other positive root of each side of an inequality preserves the direction of the inequality.) Using this fact, we get that taking the positive fourth root of both sides of $\left(1+\frac{r}{400}\right)^4 \geq 1.05$ yields $1+\frac{r}{400} \geq \sqrt[4]{1.05}$, which simplifies to $r \geq 400\left(\sqrt[4]{1.05}-1\right)$.

To compute the fourth root of 1.05, recall that for any number $x \geq 0$, $\sqrt[4]{x} = \sqrt{\sqrt{x}}$. This allows us to compute a fourth root 1.05 by taking the square root of 1.05 and then take the square root of the result.

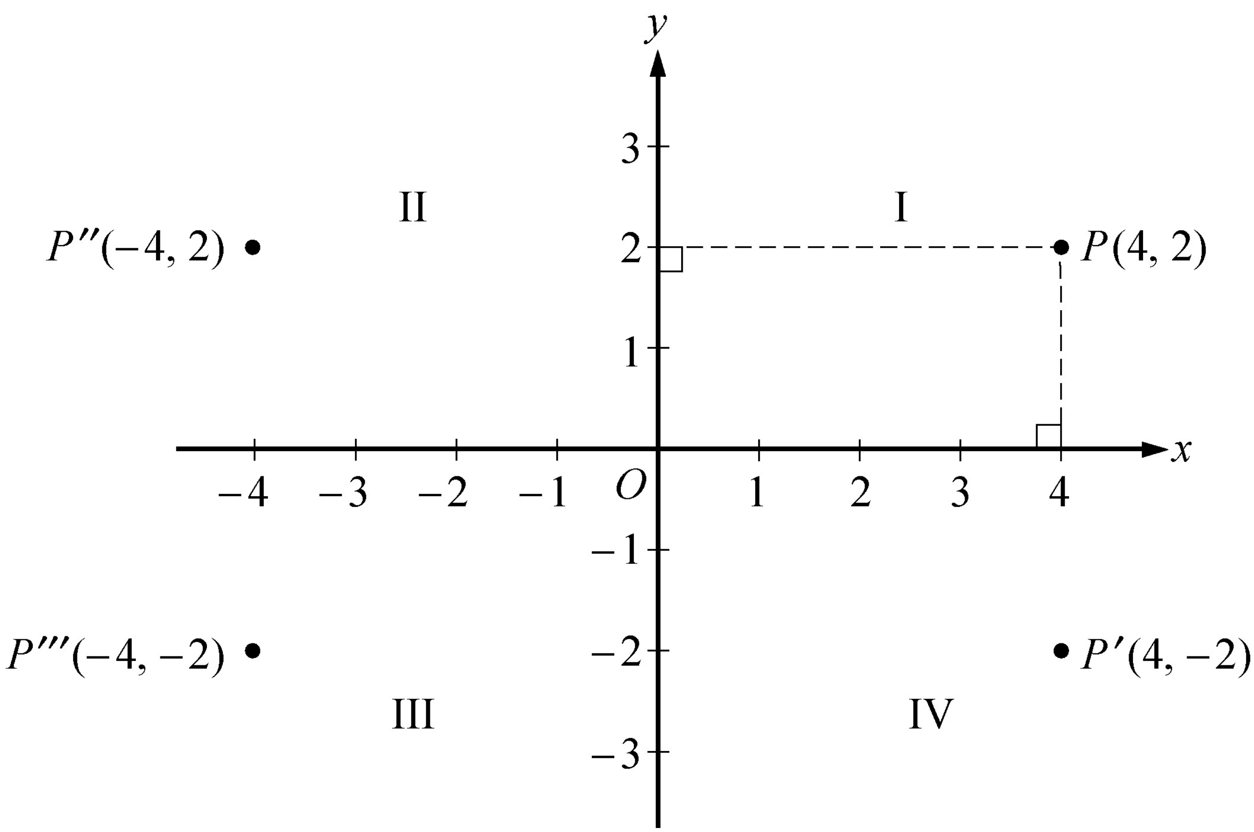
Therefore we can conclude that

$$400\left(\sqrt[4]{1.05}-1\right) = 400\left(\sqrt{\sqrt{1.05}}-1\right) \approx 4.91.$$

Since $r \geq 400\left(\sqrt[4]{1.05} - 1\right)$ and $400\left(\sqrt[4]{1.05} - 1\right)$ is approximately 4.91, the least annual interest rate is approximately 4.91 percent.

2.8 Coordinate Geometry

Two real number lines that are perpendicular to each other and that intersect at their respective zero points define a **rectangular coordinate system**, often called the ***xy*-coordinate system** or ***xy*-plane**. The horizontal number line is called the ***x*-axis** and the vertical number line is called the ***y*-axis**. The point where the two axes intersect is called the **origin**, denoted by *O*. The positive half of the *x*-axis is to the right of the origin, and the positive half of the *y*-axis is above the origin. The two axes divide the plane into four regions called **quadrants I, II, III, and IV**, as shown in Algebra Figure 1 below.



Algebra Figure 1

Each point P in the xy-plane can be identified with an ordered pair (x, y) of real numbers and is denoted by $P(x, y)$. The first number is called the ***x*-coordinate**, and the second number is called the ***y*-coordinate**. A point with coordinates (x, y) is located $|x|$ units to the right of the y-axis if x is positive or to the left of the y-axis if x is negative. Also, the point is located $|y|$ units above the x-axis if y is positive or below the x-axis if y is negative. If $x = 0$, the point lies on the y-axis, and if $y = 0$, the point lies on the x-axis. The origin has coordinates $(0, 0)$. Unless otherwise noted, the units used on the x-axis and the y-axis are the same.

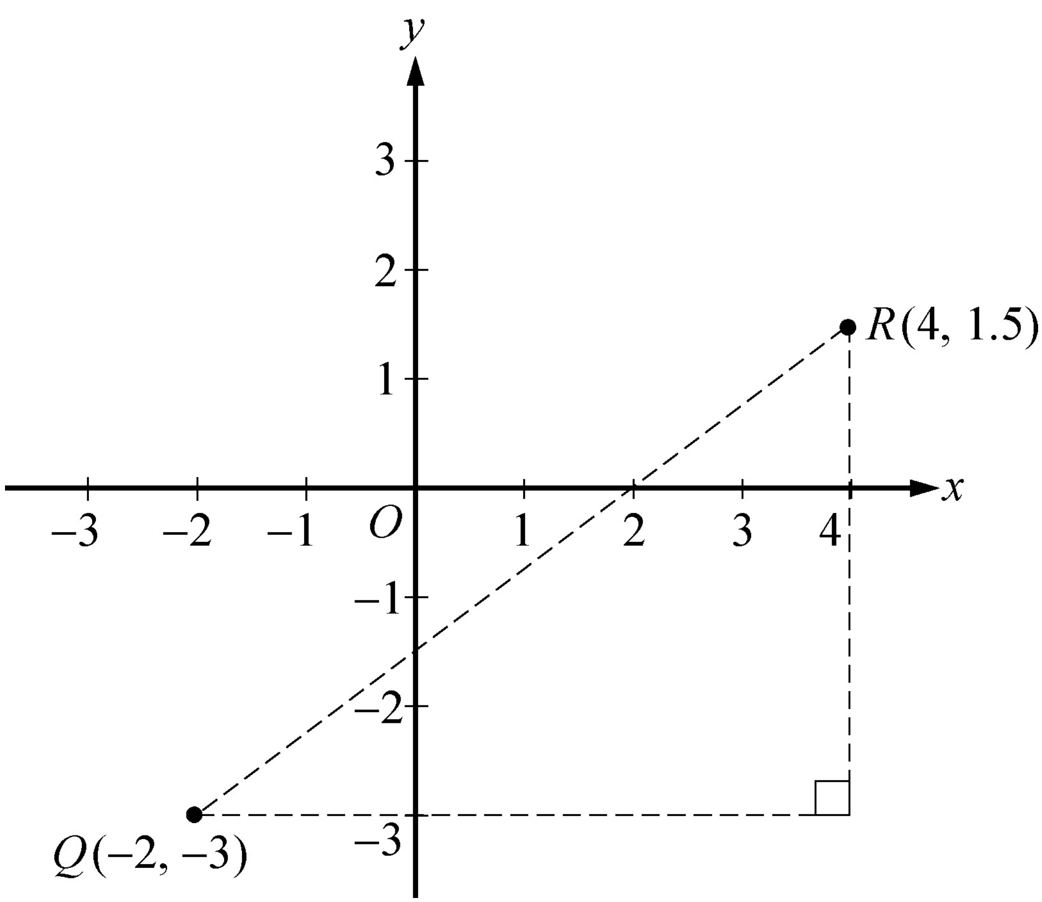
In Algebra Figure 1 above, the point $P(4, 2)$ is 4 units to the right of the y-axis and 2 units above the x-axis, the point $P'(4, -2)$ is 4 units to the right of the y-axis and 2 units below the x-axis, the point $P''(-4, 2)$ is 4 units to the left of the y-axis and 2 units above the x-axis, and the point $P'''(-4, -2)$ is 4 units to the left of the y-axis and 2 units below the x-axis.

Note that the three points $P'(4, -2)$, $P''(-4, 2)$, and $P'''(-4, -2)$ have the same coordinates as P except for the sign. These points are geometrically related to P as follows.

- P' is the **reflection of *P* about the *x*-axis**, or P' and P are **symmetric about the *x*-axis**.
- P'' is the **reflection of *P* about the *y*-axis**, or P'' and P are **symmetric about the *y*-axis**.
- P''' is the **reflection of *P* about the origin**, or P''' and P are **symmetric about the origin**.

The distance between two points in the *xy*-plane can be found by using the Pythagorean theorem. For example, the distance between the two points $Q(-2, -3)$ and $R(4, 1.5)$ in

Algebra Figure 2 below is the length of line segment QR. To find this length, construct a right triangle with hypotenuse QR by drawing a vertical line segment downward from R and a horizontal line segment rightward from Q until these two line segments intersect at the point with coordinates $(4, -3)$ forming a right angle, as shown in Algebra Figure 2. Then note that the horizontal side of the triangle has length $4 - (-2) = 6$ and the vertical side of the triangle has length $1.5 - (-3) = 4.5$.



Algebra Figure 2

Since line segment QR is the hypotenuse of the triangle, you can apply the Pythagorean theorem:

$$QR = \sqrt{6^2 + 4.5^2} = \sqrt{56.25} = 7.5.$$

(For a discussion of right triangles and the Pythagorean theorem, see Chapter 3: Geometry, Section 3.3.)

Equations in two variables can be represented as graphs in the coordinate plane. In the xy-plane, the **graph of an equation** in the variables x and y is the set of all points whose ordered pairs (x, y) satisfy the equation.

The graph of a linear equation of the form $y = mx + b$ is a straight line in the xy-plane, where m is called the **slope** of the line and b is called the ***y*-intercept.**

The ***x*-intercepts** of a graph are the x-values of the points at which the graph intersects the x-axis. Similarly, the ***y*-intercepts** of a graph are the y-values of the points at which the graph intersects the y-axis.

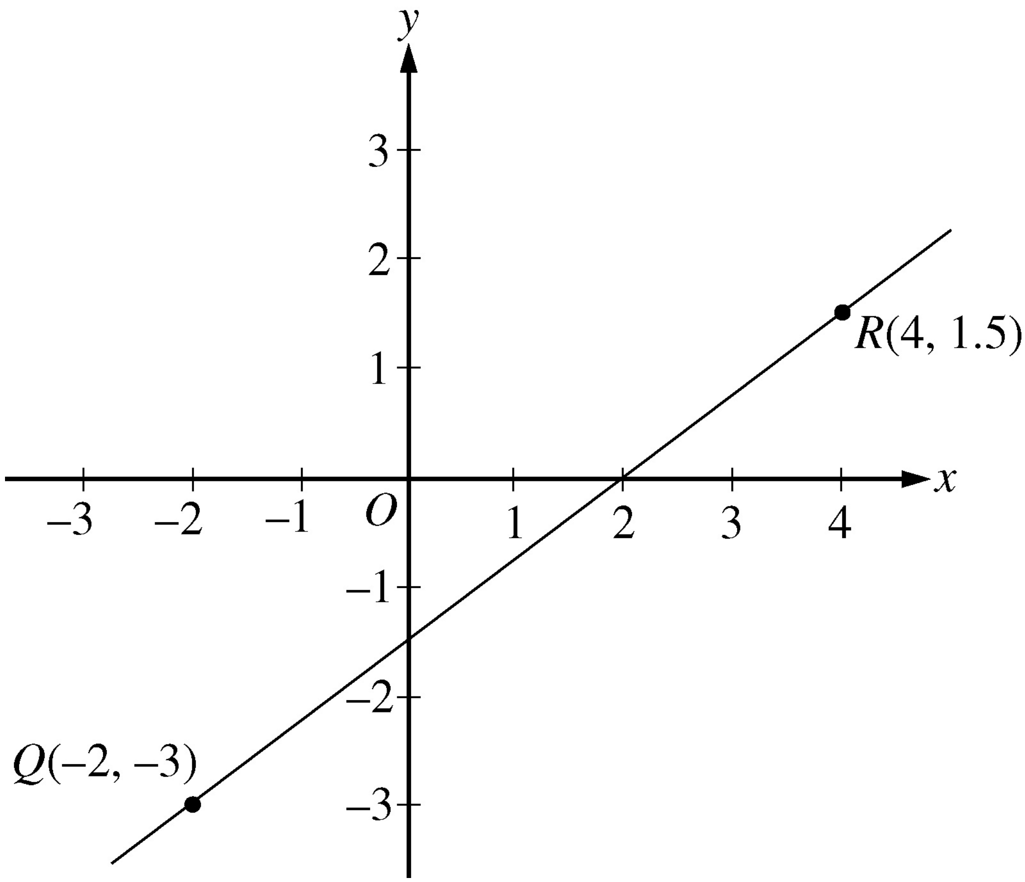
The slope of a line passing through two points $Q(x_1, y_1)$ and $R(x_2, y_2)$, where $x_1 \neq x_2$, is defined as

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

This ratio is often called "rise over run," where **rise** is the change in y when moving from Q to R and **run** is the change in x when moving from Q to R. A horizontal line has a slope of 0, since the rise is 0 for any two points on the line. So the equation of every horizontal line has the form $y = b$, where b is the y-intercept. The slope of a vertical line is not defined, since the run is 0. The equation of every vertical line has the form $x = a$, where a is the x-intercept.

Two lines are **parallel** if their slopes are equal. Two lines are **perpendicular** if their slopes are negative reciprocals of each other. For example, the line with equation $y = 2x + 5$ is perpendicular to the line with equation $y = -\frac{1}{2}x + 9$.

Example 2.8.1: Algebra Figure 3 shows the graph of the line through points $Q(-2, -3)$ and $R(4, 1.5)$ in the xy-plane



Algebra Figure 3

In Algebra Figure 3 above, the slope of the line passing through the points $Q(-2, -3)$ and $R(4, 1.5)$ is

$$\frac{1.5 - (-3)}{4 - (-2)} = \frac{4.5}{6} = 0.75.$$

Line *QR* appears to intersect the *y*-axis close to the point $(0, -1.5)$, so the *y*-intercept of the line must be close to -1.5. To get the exact value of the *y*-intercept, substitute the coordinates of any point on the line into the equation $y = 0.75x + b$, and solve it for b.

For example, if you pick the point $Q(-2, -3)$, and substitute its coordinates into the equation you get $-3 = (0.75)(-2) + b$. Then adding $(0.75)(2)$ to both sides of the equation yields $b = -3 + (0.75)(2)$, or $b = -1.5$

Therefore, the equation of line *QR* is $y = 0.75x - 1.5$.

You can see from the graph in Algebra Figure 3 that the *x*-intercept of line *QR* is 2, since *QR* passes through the point $(2, 0)$. More generally, you can find the *x*-intercept of a line by setting $y = 0$ in an equation of the line and solving it for *x*. So you can find the *x*-intercept of line *QR* by setting $y = 0$ in the equation $y = 0.75x - 1.5$ and solving it for *x*.

Setting $y = 0$ in the equation $y = 0.75x - 1.5$ gives the equation $0 = 0.75x - 1.5$. Then adding 1.5 to both sides yields $1.5 = 0.75x$. Finally, dividing both sides by 0.75 yields
$x = \frac{1.5}{0.75} = 2$.

Graphs of linear equations can be used to illustrate solutions of systems of linear equations and inequalities, as can be seen in examples 2.8.2 and 2.8.3.

Example 2.8.2: Consider the system of two linear equations in two variables:

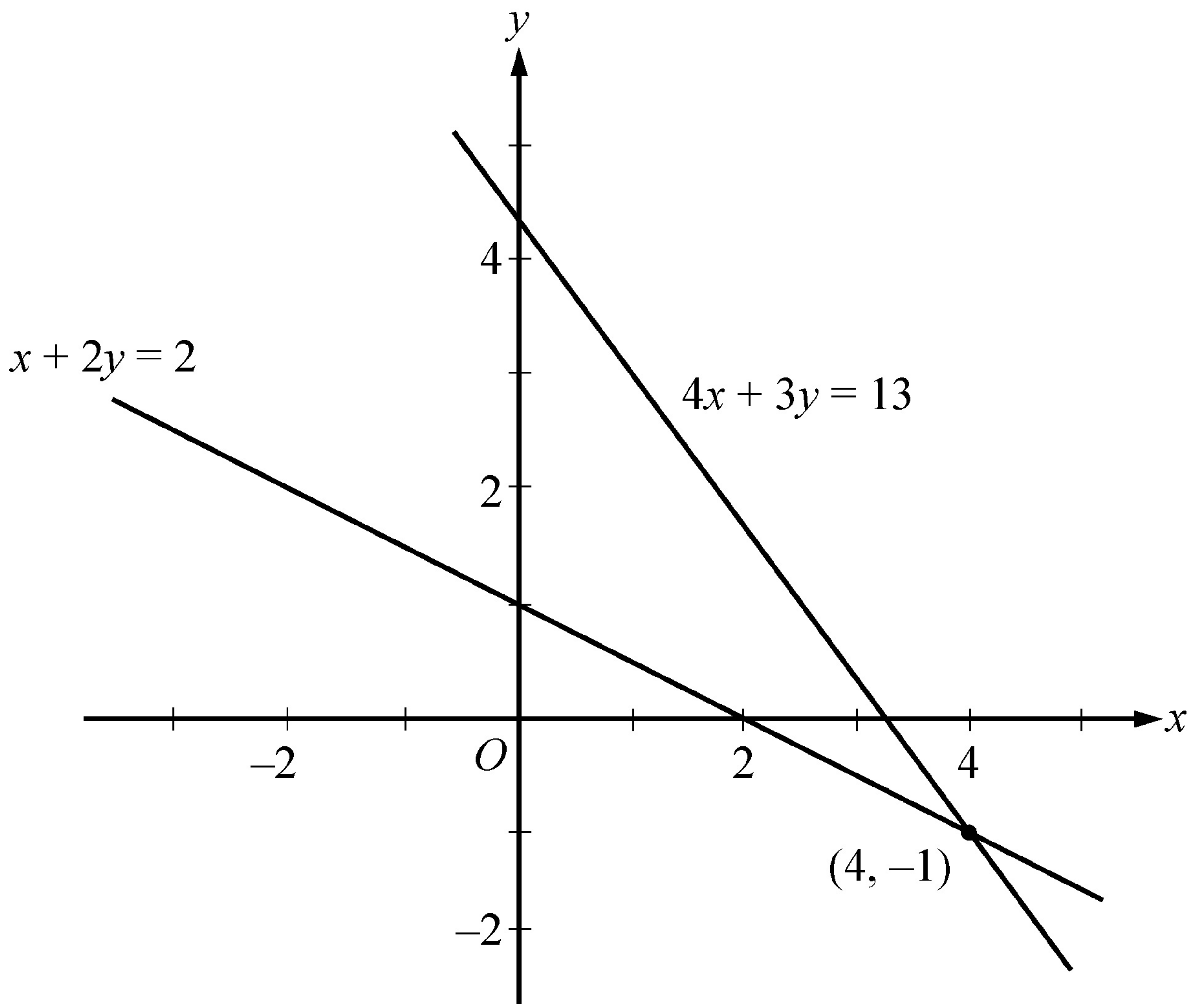
$$\begin{aligned} 4x + 3y &= 13 \\ x + 2y &= 2 \end{aligned}$$

(Recall that this system was solved by substitution, and by elimination in Chapter 2: Algebra, Section 2.3.)

Solving each equation for y in terms of x yields

$$y = -\frac{4}{3}x + \frac{13}{3}$$
$$y = -\frac{1}{2}x + 1$$

Algebra Figure 4 below shows the graphs of the two equations in the xy-plane. The solution of the system of equations is the point at which the two graphs intersect, which is $(4, -1)$.



Algebra Figure 4

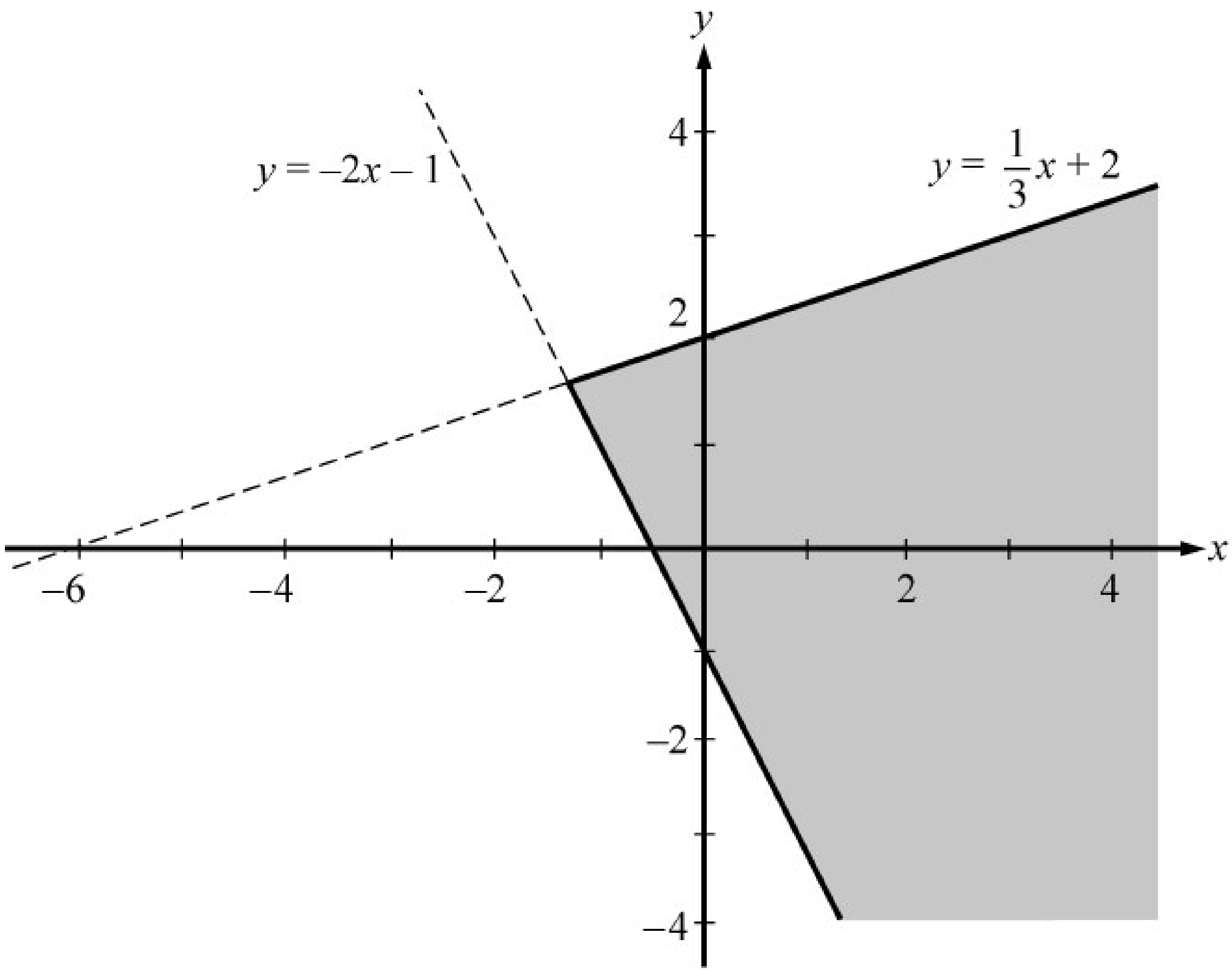
Example 2.8.3: Consider the following system of two linear inequalities.

$$x - 3y \geq -6$$
$$2x + y \geq -1$$

Solving each inequality for y in terms of x yields

$$y \leq \frac{1}{3}x + 2$$
$$y \geq -2x - 1$$

Each point (x, y) that satisfies the first inequality $y \leq \frac{1}{3}x + 2$ is either on the line $y = \frac{1}{3}x + 2$ or ***below*** the line because the y-coordinate is either equal to or ***less than*** $\frac{1}{3}x + 2$. Therefore, the graph of $y \leq \frac{1}{3}x + 2$ consists of the line $y = \frac{1}{3}x + 2$ and the entire region below it. Similarly, the graph of $y \geq -2x - 1$ consists of the line $y = -2x - 1$ and the entire region ***above*** it. Thus, the solution set of the system of inequalities consists of all of the points that lie in the shaded region shown in Algebra Figure 5 below, which is the intersection of the two regions described.

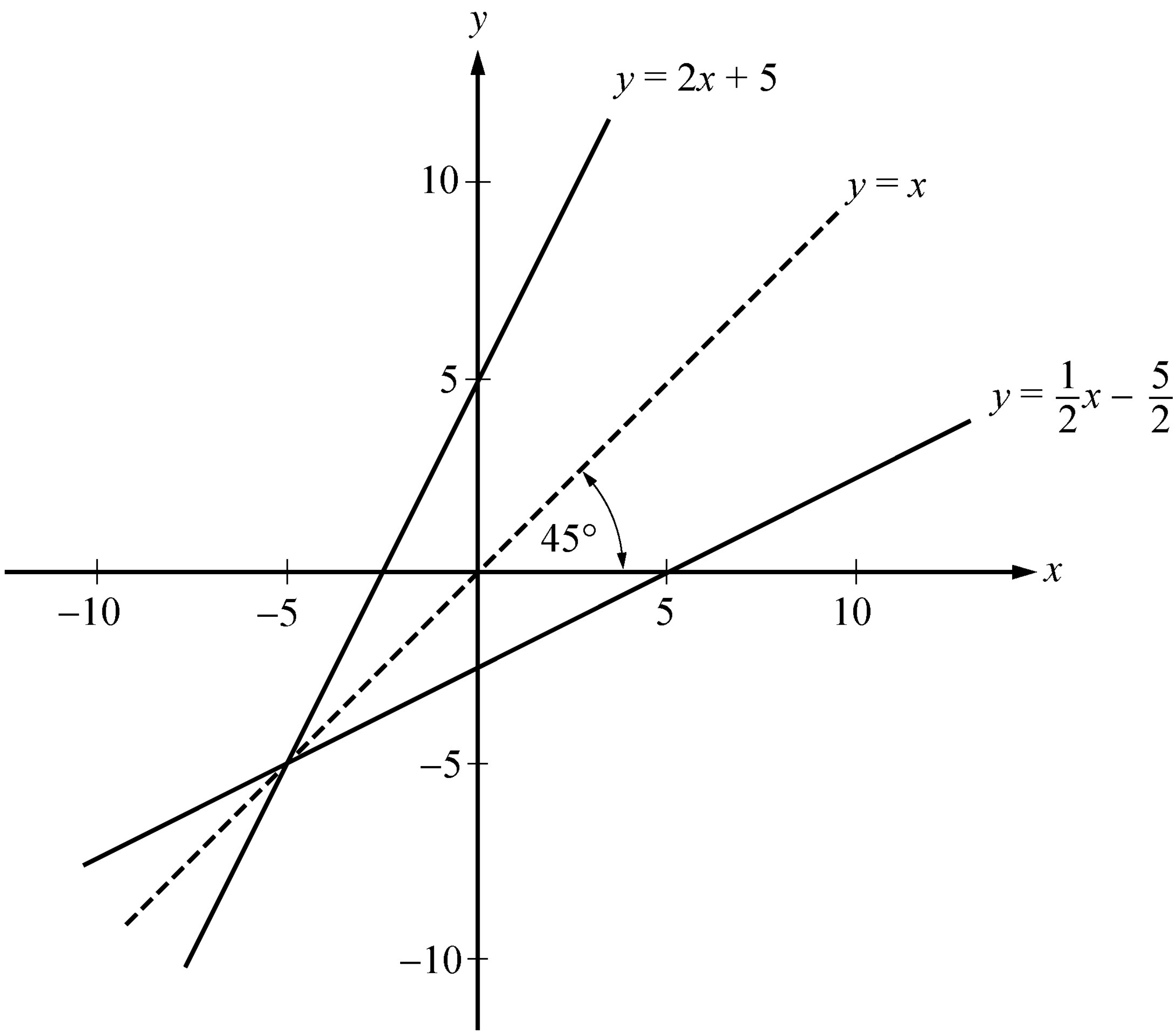


Algebra Figure 5

Symmetry with respect to the x-axis, the y-axis, and the origin is mentioned earlier in this section. Another important symmetry is symmetry with respect to the line with equation $y = x$. The line $y = x$ passes through the origin, has a slope of 1, and makes a 45-degree angle with each axis. For any point with coordinates (a, b), the point with interchanged coordinates (b, a) is the reflection of (a, b) about the line $y = x$; that is, (a, b) and (b, a) are symmetric about the line $y = x$. It follows that interchanging x and y in the equation of any graph yields another graph that is the reflection of the original graph about the line $y = x$.

Example 2.8.4: Consider the line whose equation is $y = 2x + 5$. Interchanging x and y in the equation yields $x = 2y + 5$. Solving this equation for y yields $y = \frac{1}{2}x - \frac{5}{2}$.

The line $y = 2x + 5$ and its reflection $y = \frac{1}{2}x - \frac{5}{2}$ are graphed in Algebra Figure 6 below.

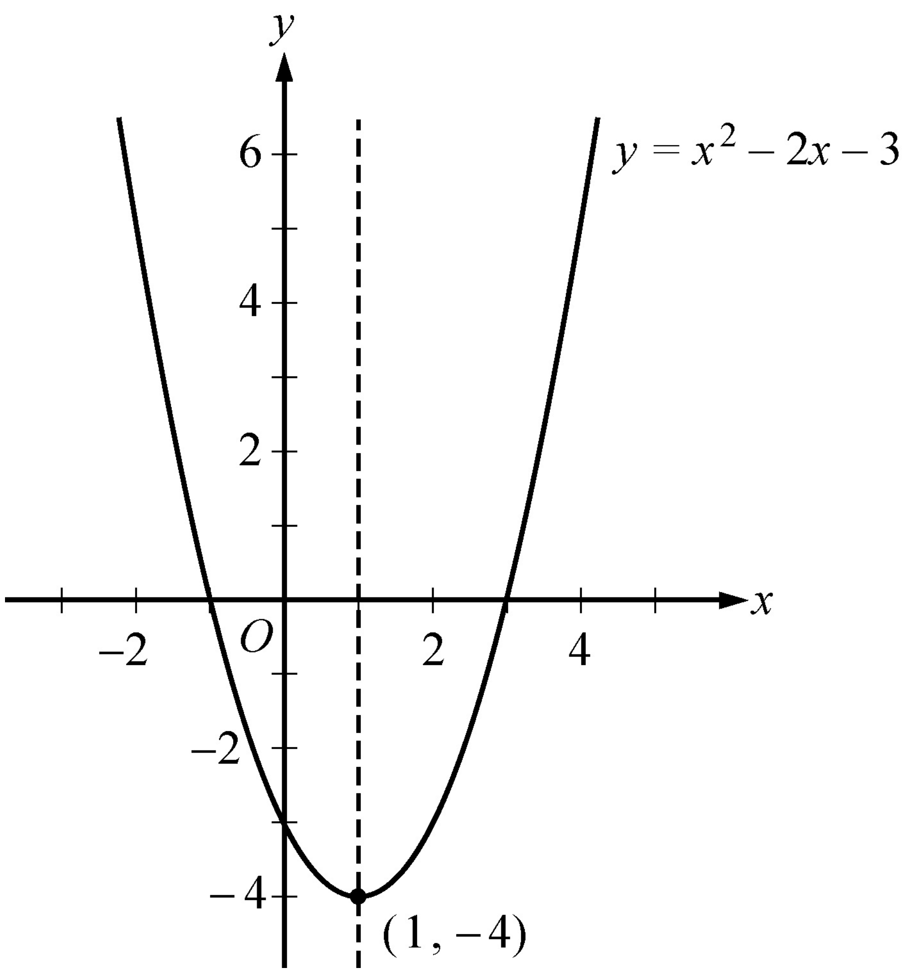


Algebra Figure 6

The line $y = x$ is a **line of symmetry** for the graphs of $y = 2x + 5$ and $y = \frac{1}{2}x - \frac{5}{2}$.

The graph of a quadratic equation of the form $y = ax^2 + bx + c$, where a, b, and c are constants and $a \neq 0$, is a **parabola**, which is a U-shaped curve. The x-intercepts of the parabola are the solutions of the equation $ax^2 + bx + c = 0$. If a is positive, the parabola opens upward and the **vertex** is its lowest point. If a is negative, the parabola opens downward and the vertex is the highest point. Every parabola is symmetric with itself about the vertical line that passes through its vertex. In particular, the two x-intercepts are equidistant from this line of symmetry.

Example 2.8.5: Consider the equation $y = x^2 - 2x - 3$. The graph of this equation is a parabola that opens upward. The x-intercepts of the parabola are -1 and 3. The values of the x-intercepts can be confirmed by solving the quadratic equation $x^2 - 2x - 3 = 0$ to get $x = -1$ and $x = 3$. The point $(1, -4)$ is the vertex of the parabola, and the line $x = 1$ is its line of symmetry. The parabola, along with its line of symmetry, is shown in Algebra Figure 7 below.



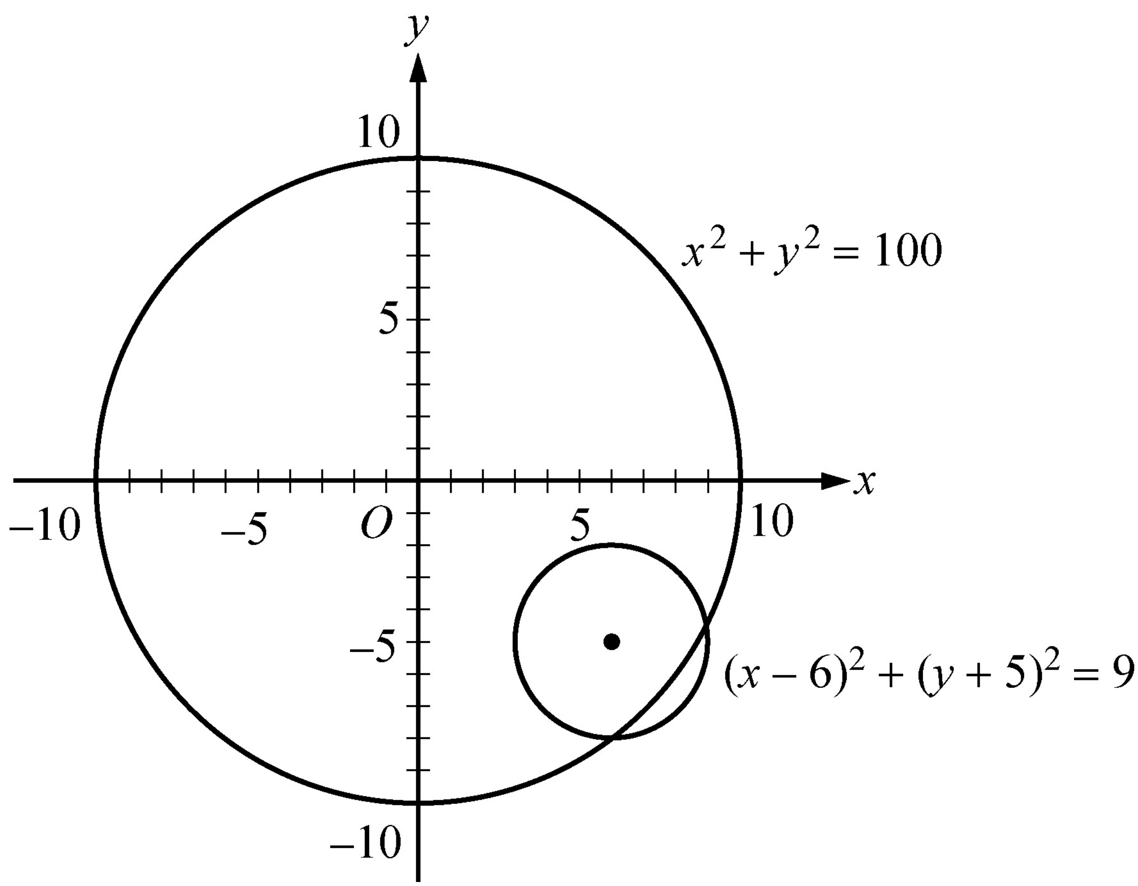
Algebra Figure 7

The y-intercept is the y-coordinate of the point on the parabola at which $x = 0,$ which is $y = 0^2 - 2(0) - 3 = -3.$

The graph of an equation of the form $(x - a)^2 + (y - b)^2 = r^2$ is a **circle** with its center at the point (a, b) and with radius r.

Example 2.8.6: Algebra Figure 8 below shows the graph of two circles in the *xy*-plane. The larger of the two circles is centered at the origin and has radius 10, so its equation is $x^2 + y^2 = 100$. The smaller of the two circles has center (6, –5) and radius 3, so its equation is

$(x - 6)^2 + (y + 5)^2 = 9$.



Algebra Figure 8

2.9 Graphs of Functions

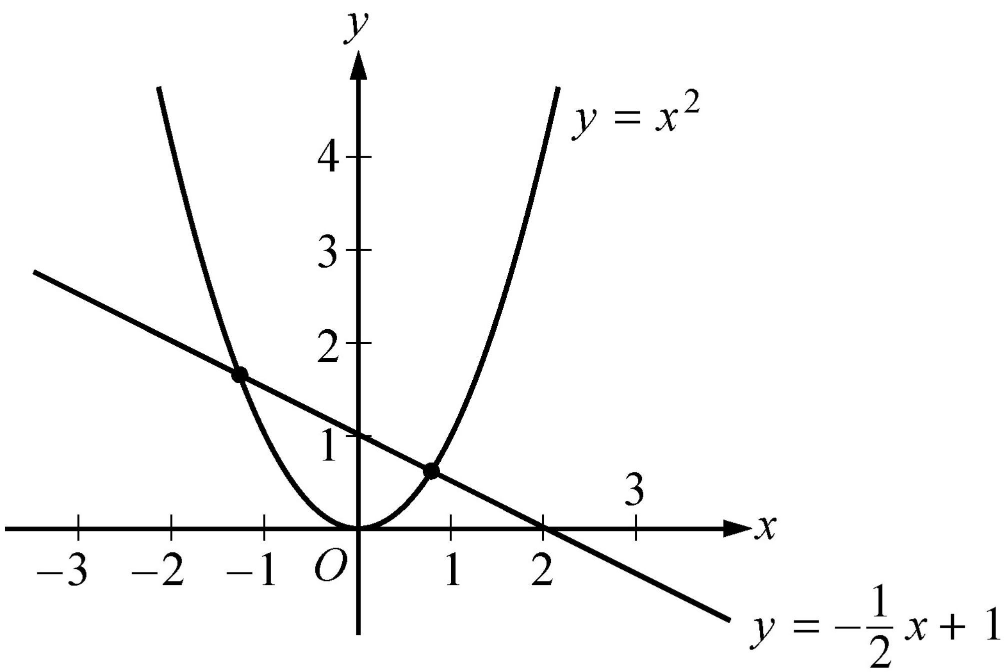
The coordinate plane can be used for graphing functions. To graph a function in the *xy*-plane, you represent each input x and its corresponding output $f(x)$ as a point (x, y), where $y = f(x)$. In other words, you use the *x*-axis for the input and the *y*-axis for the output.

Below are several examples of graphs of elementary functions.

Example 2.9.1: Consider the linear function defined by $f(x) = -\frac{1}{2}x + 1$. Its graph in the *xy*-plane is the line with the linear equation $y = -\frac{1}{2}x + 1$.

Example 2.9.2: Consider the quadratic function defined by $g(x) = x^2$. The graph of g is the parabola with the quadratic equation $y = x^2$.

The graph of both the linear equation $y = -\frac{1}{2}x + 1$ and the quadratic equation $y = x^2$ are shown in Algebra Figure 9 below.



Algebra Figure 9

Note that the graphs *f* and *g* in Algebra Figure 9 above intersect at two points. These are the points at which $g(x) = f(x)$. We can find these points algebraically as follows.

Set $g(x) = f(x)$ and get $x^2 = -\frac{1}{2}x + 1$, which is equivalent to $x^2 + \frac{1}{2}x - 1 = 0$; or $2x^2 + x - 2 = 0$.

Then solve the equation $2x^2 + x - 2 = 0$ for x using the quadratic formula and get $x = \dfrac{-1 \pm \sqrt{1+16}}{4}$,

which represents the x-coordinates of the two solutions

$$x = \frac{-1+\sqrt{17}}{4} \approx 0.78 \qquad \text{and} \qquad x = \frac{-1-\sqrt{17}}{4} \approx -1.28.$$

With these input values, the corresponding y-coordinates can be found using either f or g:

$$g\left(\frac{-1+\sqrt{17}}{4}\right) = \left(\frac{-1+\sqrt{17}}{4}\right)^2 \approx 0.61 \quad \text{and}$$

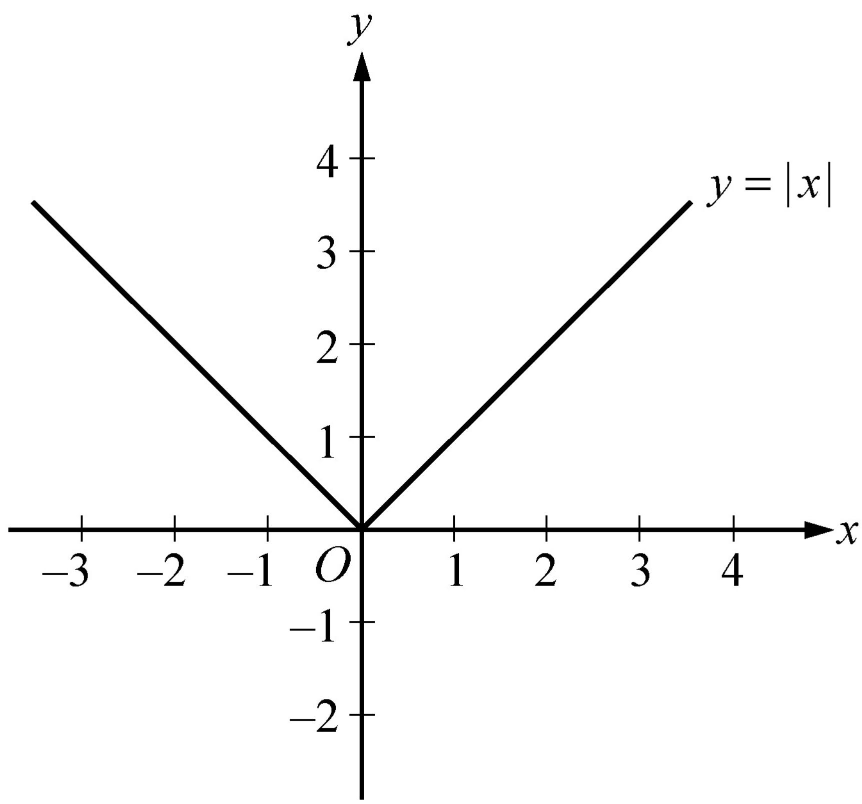
$$g\left(\frac{-1-\sqrt{17}}{4}\right) = \left(\frac{-1-\sqrt{17}}{4}\right)^2 \approx 1.64.$$

Thus, the two intersection points can be approximated by (0.78, 0.61) and (–1.28, 1.64).

Example 2.9.3: Consider the absolute value function defined by $h(x) = |x|$. By using the definition of absolute value (see Chapter 1: Arithmetic, Section 1.5), h can be expressed as a **piecewise-defined** function:

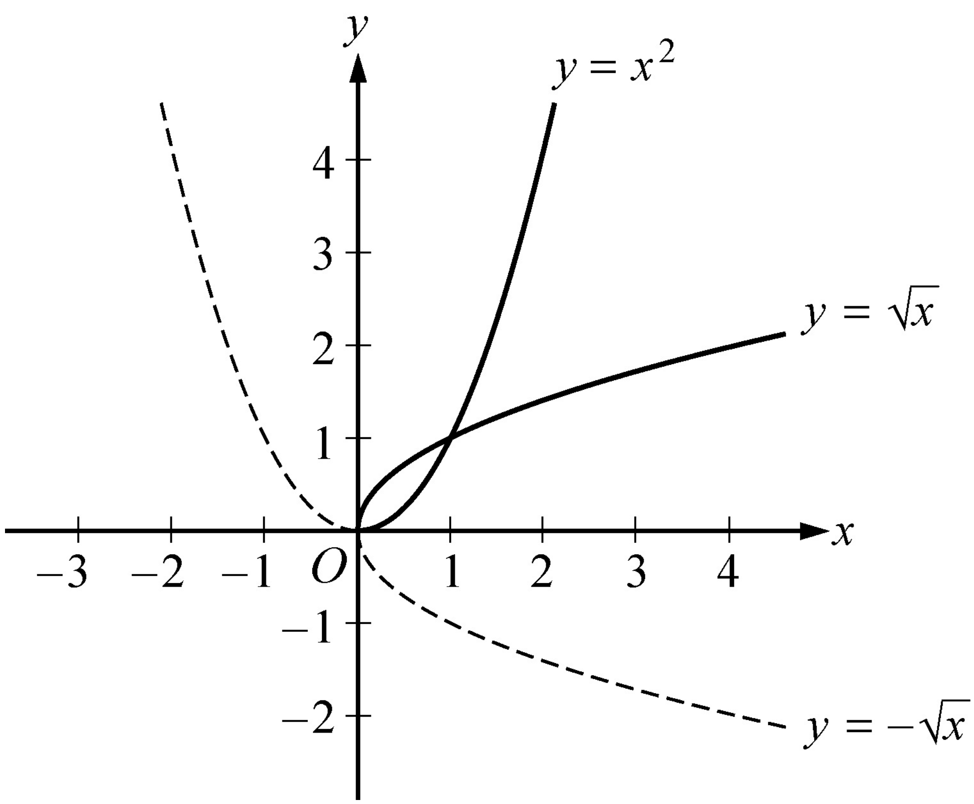
$$h(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

The graph of this function is V-shaped and consists of two linear pieces, $y = x$ and $y = -x$, joined at the origin, as shown in Algebra Figure 10 below.



Algebra Figure 10

Example 2.9.4: This example is based on Algebra Figure 11 below, which is the graph of 2 parabolas.



Algebra Figure 11

One of the parabolas is $y = x^2$, which opens upward, and the other looks like the parabola $y = x^2$, but instead of opening upward, it opens to the right. The vertex of both parabolas is at the origin.

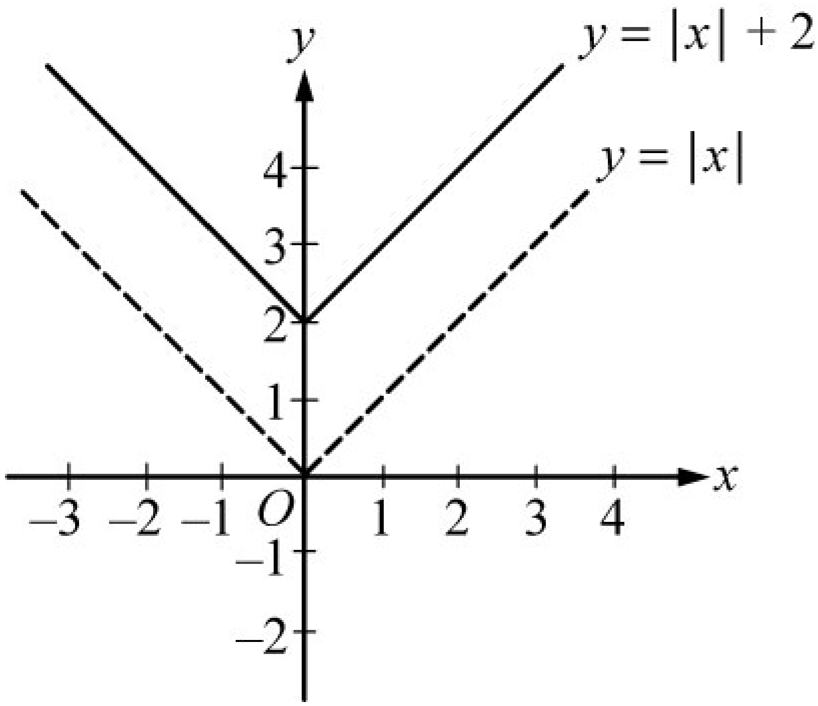
Consider the positive square-root function defined by $j(x) = \sqrt{x}$ for $x \geq 0$. The graph of this function is the upper half of the right facing parabola in Algebra Figure 11, that is, the solid part of the parabola, the part above the *x*-axis. Also consider the negative square-root function defined by $k(x) = -\sqrt{x}$ for $x \geq 0$. The graph of this function is the bottom half of the right facing parabola; that is the dashed part of the parabola, the part below the *x*-axis.

The graphs of $y = \sqrt{x}$ and $y = -\sqrt{x}$ are halves of a parabola because they are reflections of the right and left halves, respectively, of the parabola $y = x^2$ about the line $y = x$. This follows from squaring both sides of the two square root equations to get $y^2 = x$ and then interchanging *x* and *y* to get $y = x^2$.

Also note that $y = -\sqrt{x}$ is the reflection of $y = \sqrt{x}$ about the *x*-axis. In general, for any function *h*, the graph of $y = -h(x)$ is the **reflection** of the graph of $y = h(x)$ about the *x*-axis.

Example 2.9.5: Consider the functions defined by $f(x) = |x| + 2$ and $g(x) = (x + 1)^2$. These functions are related to the absolute value function $|x|$ and the quadratic function x^2, respectively, in simple ways.

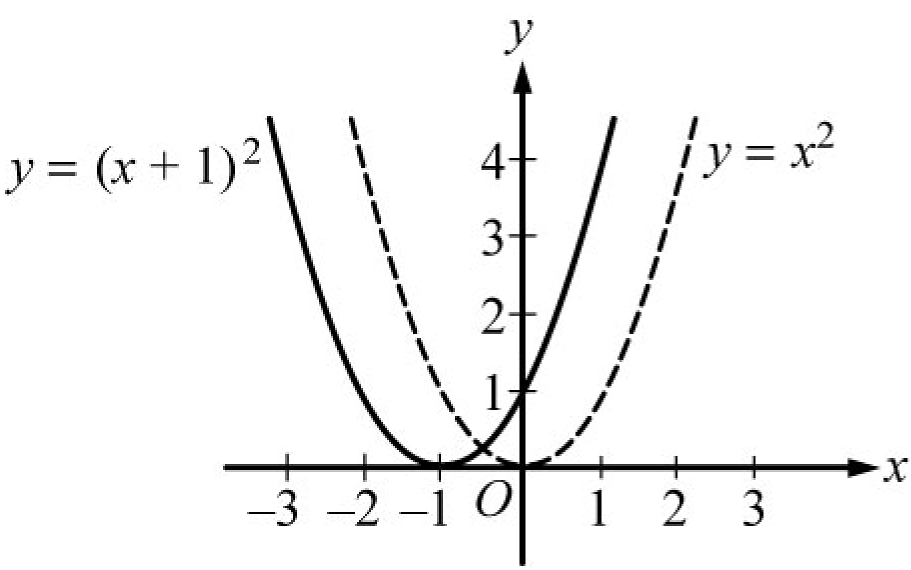
The graph of $f(x) = |x| + 2$ is the graph of $y = |x|$ shifted upward by 2 units. Recall that the graph of $y = |x|$ is a V-shaped curve. Therefore, the graph of $f(x) = |x| + 2$ is also a V-shaped curve. The graphs of both of these functions is shown in Algebra Figure 12.



Algebra Figure 12

Similarly, the graph of the function $k(x) = |x| - 5$ is the graph of $y = |x|$ shifted downward by 5 units. (The graphs of these functions are not shown).

The graph of $g(x) = (x + 1)^2$ is the graph of $y = x^2$ shifted to the left by 1 unit, as shown in the Algebra Figure 13 below.



Algebra Figure 13

Similarly, the graph of the function $j(x) = (x - 4)^2$ is the graph of $y = x^2$ shifted to the right by 4 units. (The graphs of these functions are not shown.) To double-check the direction of the shift, you can plot some corresponding values of the original function and the shifted function.

In general, for any function $h(x)$ and any positive number c, the following are true.

The graph of $h(x) + c$ is the graph of $h(x)$ **shifted upward** by c units.

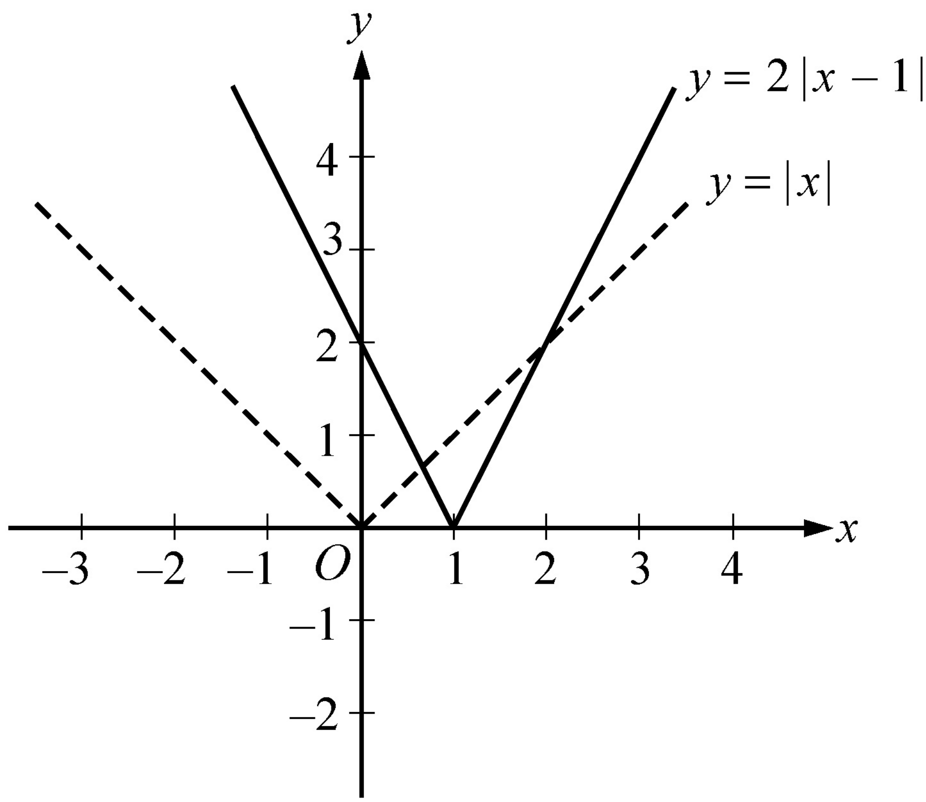
The graph of $h(x) - c$ is the graph of $h(x)$ **shifted downward** by c units.

The graph of $h(x + c)$ is the graph of $h(x)$ **shifted to the left** by c units.

The graph of $h(x - c)$ is the graph of $h(x)$ **shifted to the right** by c units.

Example 2.9.6: Consider the functions defined by $f(x) = 2|x-1|$ and $g(x) = -\frac{x^2}{4}$. These functions are related to the absolute value function $|x|$ and the quadratic function x^2, respectively, in more complicated ways than in the preceding example.

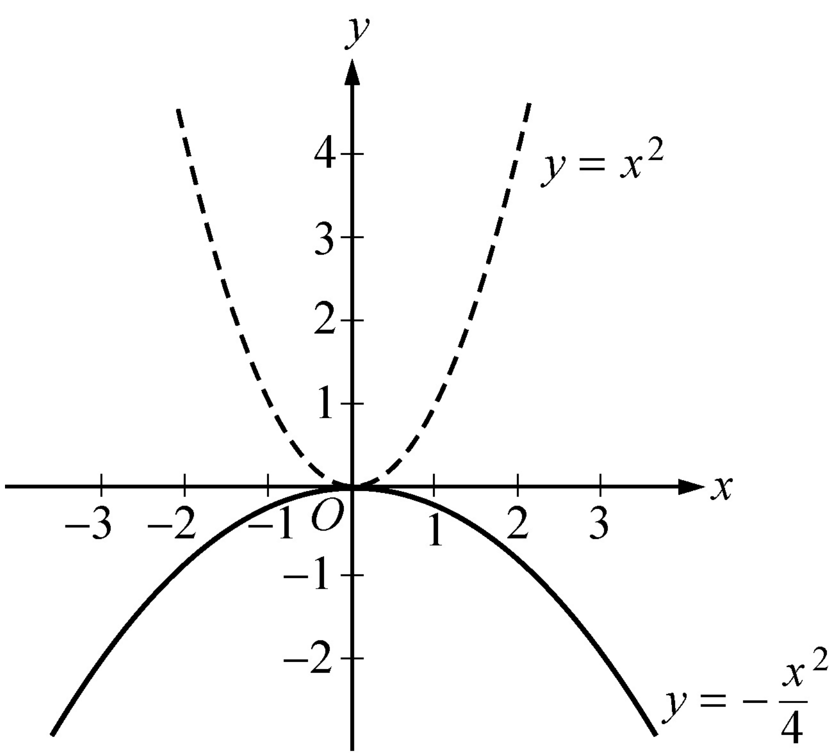
The graph of $f(x) = 2|x - 1|$ is the graph of $y = |x|$ shifted to the right by 1 unit and then stretched, or dilated, vertically away from the x-axis by a factor of 2, as shown in the Algebra Figure 14 below.



Algebra Figure 14

Similarly, the graph of the function $\frac{1}{2}|x - 1|$ is the graph of $|x|$ shifted to the right by 1 unit and then shrunk, or contracted, vertically toward the x-axis by a factor of $\frac{1}{2}$ (the graphs of these functions are not shown).

The graph of $g(x) = -\frac{x^2}{4}$ is the graph of $y = x^2$ contracted vertically toward the x-axis by a factor of $\frac{1}{4}$ and then reflected in the x-axis, as shown in Algebra Figure 15 below.



Algebra Figure 15

In general, for any function $h(x)$ and any positive number c, the following are true.

The graph of $c\,h(x)$ is the graph of $h(x)$ **stretched vertically** by a factor of c if $c > 1$.

The graph of $c\,h(x)$ is the graph of $h(x)$ **shrunk vertically** by a factor of c if $0 < c < 1$.

Algebra Exercises

1. Find an algebraic expression to represent each of the following.

(a) The square of y is subtracted from 5, and the result is multiplied by 37.

(b) Three times x is squared, and the result is divided by 7.

(c) The product of $(x + 4)$ and y is added to 18.

2. Simplify each of the following algebraic expressions.

(a) $3x^2 - 6 + x + 11 - x^2 + 5x$

(b) $3(5x - 1) - x + 4$

(c) $\dfrac{x^2 - 16}{x - 4}$, where $x \neq 4$

(d) $(2x + 5)(3x - 1)$

3.

(a) What is the value of $f(x) = 3x^2 - 7x + 23$ when $x = -2$?

(b) What is the value of $h(x) = x^3 - 2x^2 + x - 2$
when $x = 2$?

(c) What is the value of $k(x) = \frac{5}{3}x - 7$ when $x = 0$?

4. If the function g is defined for all nonzero numbers y by $g(y) = \frac{y}{|y|}$, find the value of each of the following.

(a) $g(2)$

(b) $g(-2)$

(c) $g(2) - g(-2)$

5. Use the rules of exponents to simplify the following.

(a) $\left(n^{5}\right)\left(n^{-3}\right)$

(b) $\left(s^{7}\right)\left(t^{7}\right)$

(c) $\dfrac{r^{12}}{r^{4}}$

(d) $\left(\dfrac{2a}{b}\right)^{5}$

(e) $\left(w^{5}\right)^{-3}$

(f) $\left(5^{0}\right)\left(d^{3}\right)$

(g) $\dfrac{\left(x^{10}\right)\left(y^{-1}\right)}{\left(x^{-5}\right)\left(y^{5}\right)}$

(h) $\left(\dfrac{3x}{y}\right)^{2} \div \left(\dfrac{1}{y}\right)^{5}$

6. Solve each of the following equations for x.

(a) $5x - 7 = 28$

(b) $12 - 5x = x + 30$

(c) $5(x + 2) = 1 - 3x$

(d) $(x + 6)(2x - 1) = 0$

(e) $x^2 + 5x - 14 = 0$

(f) $x^2 - x - 1 = 0$

7. Solve each of the following systems of equations for x and y.

(a)
$$x + y = 24$$
$$x - y = 18$$

(b)
$$3x - y = -5$$
$$x + 2y = 3$$

(c)
$$15x - 18 - 2y = -3x + y$$
$$10x + 7y + 20 = 4x + 2$$

8. Solve each of the following inequalities for x.

(a) $-3x > 7 + x$

(b) $25x + 16 \geq 10 - x$

(c) $16 + x > 8x - 12$

9. For a given two-digit positive integer, the tens digit is 5 more than the units digit. The sum of the digits is 11. Find the integer.

10. If the ratio of $2x$ to $5y$ is 3 to 4, what is the ratio of x to y ?

11. Kathleen's weekly salary was increased by 8 percent to $237.60. What was her weekly salary before the increase?

12. A theater sells children's tickets for half the adult ticket price. If 5 adult tickets and 8 children's tickets cost a total of $27, what is the cost of an adult ticket?

13. Pat invested a total of \$3,000. Part of the money was invested in a money market account that paid 10 percent simple annual interest, and the remainder of the money was invested in a fund that paid 8 percent simple annual interest. If the interest earned at the end of the first year from these investments was \$256, how much did Pat invest at 10 percent and how much at 8 percent?

14. Two cars started from the same point and traveled on a straight course in opposite directions for exactly 2 hours, at which time they were 208 miles apart. If one car traveled, on average, 8 miles per hour faster than the other car, what was the average speed of each car for the 2-hour trip?

15. A group can charter a particular aircraft at a fixed total cost. If 36 people charter the aircraft rather than 40 people, then the cost per person is greater by $12.

(a) What is the fixed total cost to charter the aircraft?

(b) What is the cost per person if 40 people charter the aircraft?

16. An antiques dealer bought c antique chairs for a total of x dollars. The dealer sold each chair for y dollars.

(a) Write an algebraic expression for the profit, P, earned from buying and selling the chairs.

(b) Write an algebraic expression for the profit <u>per chair</u>.

17. In the coordinate system in Algebra Figure 16 below, find the following.

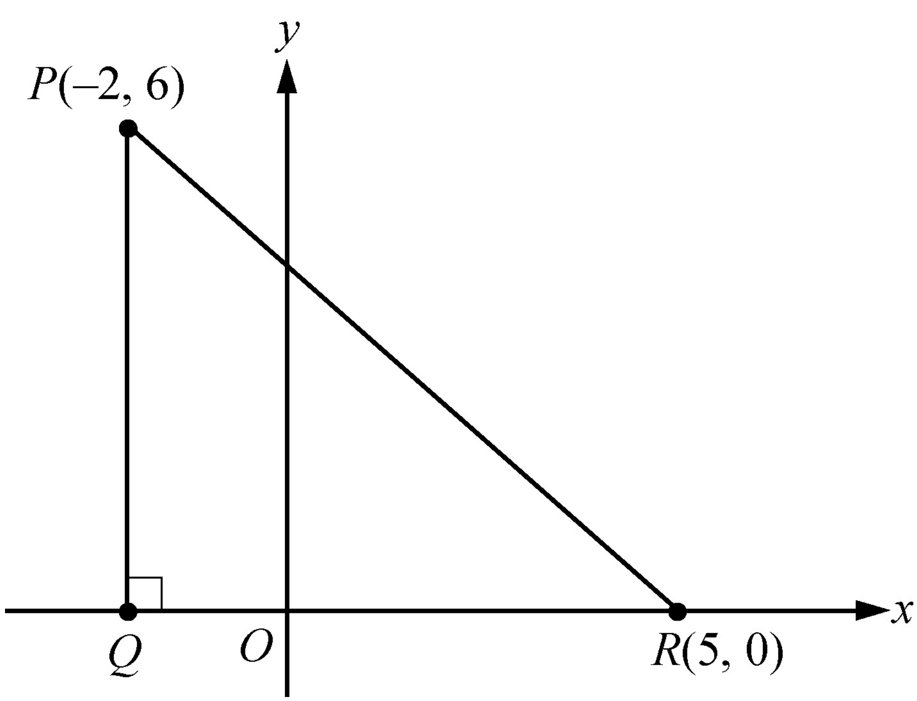
(a) Coordinates of point Q

(b) Lengths of PQ, QR, and PR

(c) Perimeter of $\triangle PQR$

(d) Area of $\triangle PQR$

(e) Slope, y-intercept, and equation of the line passing through points P and R



Algebra Figure 16

18. In the xy-plane, find the following.

(a) Slope and y-intercept of the line with equation $2y + x = 6$

(b) Equation of the line passing through the point $(3, 2)$ with y-intercept 1

(c) The y-intercept of a line with slope 3 that passes through the point $(-2, 1)$

(d) The x-intercepts of the graphs in (a), (b), and (c)

19. For the parabola $y = x^2 - 4x - 12$ in the xy-plane, find the following.

(a) The x-intercepts

(b) The y-intercept

(c) Coordinates of the vertex

20. For the circle $(x-1)^2 + (y+1)^2 = 20$ in the xy-plane, find the following.

(a) Coordinates of the center

(b) Radius

(c) Area

21. For each of the following functions, give the domain and a description of the graph $y = f(x)$ in the xy-plane, including its shape, and the x- and y-intercepts.

(a) $f(x) = -4$

(b) $f(x) = 100 - 900x$

(c) $f(x) = 5 - (x+20)^2$

(d) $f(x) = \sqrt{x+2}$

(e) $f(x) = x + |x|$

Answers to Algebra Exercises

1.

(a) $37\left(5 - y^2\right)$, or $185 - 37y^2$

(b) $\dfrac{(3x)^2}{7}$, or $\dfrac{9x^2}{7}$

(c) $18 + (x + 4)(y)$, or $18 + xy + 4y$

2.

(a) $2x^2 + 6x + 5$

(b) $14x + 1$

(c) $x + 4$

(d) $6x^2 + 13x - 5$

3.

(a) 49

(b) 0

(c) -7

4.

(a) 1

(b) -1

(c) 2

5.

(a) n^2

(b) $(st)^7$

(c) r^8

(d) $\frac{32a^5}{b^5}$

(e) $\frac{1}{w^{15}}$

(f) d^3

(g) $\frac{x^{15}}{y^6}$

(h) $9x^2y^3$

6.

(a) 7

(b) -3

(c) $-\frac{9}{8}$

(d) $-6, \frac{1}{2}$

(e) $-7, 2$

(f) $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

7.

(a) $x = 21$, $y = 3$

(b) $x = -1$, $y = 2$

(c) $x = \frac{1}{2}$, $y = -3$

8.

(a) $x < -\frac{7}{4}$

(b) $x \geq -\frac{3}{13}$

(c) $x < 4$

9. 83

10. 15 to 8

11. $220

12. $3

13. $800 at 10% and $2,200 at 8%

14. 48 mph and 56 mph

15.

(a) \$4,320

(b) \$108

16.

(a) $P = cy - x$

(b) Profit per chair: $\frac{P}{c} = \frac{cy - x}{c} = y - \frac{x}{c}$

17.

(a) $(-2, 0)$

(b) $PQ = 6,\ QR = 7,\ PR = \sqrt{85}$

(c) $13 + \sqrt{85}$

(d) 21

(e) Slope: $-\frac{6}{7}$; y-intercept: $\frac{30}{7}$;

equation of line: $y = -\frac{6}{7}x + \frac{30}{7}$, or $7y + 6x = 30$

18.

(a) Slope: $-\frac{1}{2}$; y-intercept: 3

(b) $y = \frac{x}{3} + 1$

(c) 7

(d) $6,\ -3,\ -\frac{7}{3}$

19.

(a) $x = -2$ and $x = 6$

(b) $y = -12$

(c) $(2, -16)$

20.

(a) $(1, -1)$

(b) $\sqrt{20}$

(c) 20π

21.

(a) Domain: the set of all real numbers. The graph is a horizontal line with y-intercept -4 and no x-intercept.

(b) Domain: the set of all real numbers. The graph is a line with slope -900, y-intercept 100, and x-intercept $\frac{1}{9}$.

(c) Domain: the set of all real numbers. The graph is a parabola opening downward with vertex at $(-20, 5)$, line of symmetry $x = -20$, y-intercept -395, and x-intercepts $-20 \pm \sqrt{5}$.

(d) Domain: the set of numbers greater than or equal to -2. The graph is half a parabola opening to the right with vertex at $(-2, 0)$, x-intercept -2, and y-intercept $\sqrt{2}$.

(e) Domain: the set of all real numbers. The graph is two half-lines joined at the origin: one half-line is the negative x-axis and the other is a line starting at the origin with slope 2. Every nonpositive number is an x-intercept, and the y-intercept is 0. The function is equal to the following piecewise-defined function

$$f(x) = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$